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Two-Sided Bounds for the Inverse of an *H*-Matrix

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ABSTRACT

The paper presents a simple characterization of a real H-matrix and two-sided componentwise bounds for its inverse in terms of the comparison matrix and the so-called Z- and positive parts. These bounds, improving the well-known Ostrowsky result, are also extended to a larger class of matrices.

Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $1 \leqslant i, j \leqslant n$, be an $n \times n$ H-matrix, i.e., its comparison matrix $\mathscr{M}(A) = (m_{ij})$, $1 \leqslant i, j \leqslant n$, where

$$m_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \qquad 1 \leqslant i, j \leqslant n, \tag{1}$$

be an M-matrix. Recall that a square matrix $B \in \mathbb{R}^{n\times n}$ with nonpositive off-diagonal entries is an M-matrix [1] if and only if one of the following conditions is satisfied:

(i) There exists a vector v with positive components such that the vector Bv is also positive.

(ii) B is monotone, or, equivalently, B is nonsingular and B^{-1} is nonnegative.

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(iii) $B = \alpha I - P$, P is nonnegative, and $\alpha > \rho(P)$, where $\rho(P)$ is the spectral radius of P.

The well-known Ostrowsky result [4] states that the inverse A^{-1} to an H-matrix A can be bounded as follows:

$$|A^{-1}| \leqslant \mathscr{M}(A)^{-1}, \tag{2}$$

where the matrix absolute value and the inequality are understood componentwise. The amount of overestimation in (2) was bounded by A. Neumaier in [2]. He showed that

$$|A^{-1}| \leqslant \mathscr{M}(A)^{-1} \leqslant (I + \Omega)|A^{-1}|, \tag{3}$$

where $\Omega = (I - |\Delta|)^{-1}(|\Delta| - \Delta)$ is nonnegative, $\Delta = I - \mathrm{Diag}(A)^{-1}A$, and $\mathrm{Diag}(A)$ is the diagonal matrix whose diagonal entries coincide with those of A.

A. Neumaier proved also [2] that the Ostrowsky bound (2) can be improved using a triangular decomposition $A = LU$, where the matrices L and U are respectively lower and upper triangular. Namely, componentwise,

$$|A^{-1}| \leqslant \mathscr{M}(U)^{-1}\mathscr{M}(L)^{-1} \leqslant \mathscr{M}(A)^{-1}. \tag{4}$$

In this note we derive a simple characterization of a real H-matrix A with positive diagonal entries and two-sided bounds for A^{-1} using the so-called Z-part of A and also improving the Ostrowsky bound (2). In what follows we assume without loss of generality that $a_{ii} > 0$, $i = 1, \dots, n$, which can be achieved by an appropriate diagonal scaling.

Let us now introduce some more notation. All matrix and vector inequalities below are understood componentwise. The matrix $A \in \mathbb{R}^{n \times n}$ is split as

$$A = B + P, \tag{5}$$

where B is the "Z-part" of A and P is the positive off-diagonal part of A, i.e., $B = (b_{ij})$ and $P = (p_{ij})$, $1 \leqslant i,\ j \leqslant n$, are defined by the following relations:

$$b_{ij} = \begin{cases} 0 & \text{if } a_{ij} > 0 \text{ and } i \neq j, \\ a_{ij} & \text{if } a_{ij} \leqslant 0 \text{ or } i = j; \end{cases} \tag{6}$$

$$p_{ij} = \begin{cases} 0 & \text{if } a_{ij} \leqslant 0 \text{ or } i = j, \\ a_{ij} & \text{if } a_{ij} > 0 \text{ and } i \neq j. \end{cases} \tag{7}$$

It follows from (5)–(7) and (1) that

$$\mathcal{M}(A) = B - P. \tag{8}$$

In terms of the Z- and positive parts, necessary and sufficient conditions for a matrix with positive diagonal entries to be an H-matrix are provided by the following theorem.

THEOREM 1. *A matrix $A \in \mathbb{R}^{n \times n}$ with positive diagonal entries is an H-matrix if and only if its Z-part B is an M-matrix and $\rho(B^{-1}P) < 1$, where $P = A - B$ is the positive off-diagonal part of A and $\rho(B^{-1}P)$ is the spectral radius of $B^{-1}P$.*

Proof. Assume that A is an H-matrix. Then for an M-matrix $\mathcal{M}(A)$ there exists a positive vector v such that $\mathcal{M}(A)v > 0$, implying

$$Bv = \mathcal{M}(A)v + Pv \geqslant \mathcal{M}(A)v > 0.$$

Therefore, the Z-part of an H-matrix is in fact an M-matrix. Since the matrices $\mathcal{M}(A)$ and B are M-matrices while P is nonnegative, the splitting (8) is regular and hence convergent [1], i.e., $\rho(B^{-1}P) < 1$. Conversely, if B is an M-matrix and $\rho(B^{-1}P) < 1$, then the comparison matrix

$$\mathcal{M}(A) = B(I - B^{-1}P)$$

has a nonnegative inverse, since B and $I - B^{-1}P$ are M-matrices. Thus, $\mathcal{M}(A)$ is an M-matrix, implying that A is an H-matrix. ■

Two-sided componentwise bounds for A^{-1} in terms of $\mathcal{M}(A)^{-1}$, B^{-1}, and P improving (2) are given in the next theorem.

THEOREM 2. *The inverse of a real H-matrix A with positive diagonal entries satisfies the following componentwise inequalities:*

$$-\mathcal{M}(A)^{-1} + 2B^{-1} \leqslant A^{-1} \leqslant \mathcal{M}(A)^{-1} - 2B^{-1}PB^{-1}, \tag{9}$$

where the comparison M-matrix $\mathcal{M}(A)$, the Z-part B, and the positive part P of A are defined by (1), (6), and (7) respectively.

Proof. By (5)

$$A^{-1}B = (I + B^{-1}P)^{-1},$$

where $B^{-1}P$ is a nonnegative matrix and $\rho(B^{-1}P) < 1$ by Theorem 1. Therefore

$$\begin{aligned} A^{-1}B &= (I - B^{-1}P)^{-1} - 2B^{-1}P\left[I - (B^{-1}P)^2\right]^{-1} \\ &\leqslant (I - B^{-1}P)^{-1} - 2B^{-1}P \\ &\overset{(8)}{=} \mathscr{M}(A)^{-1}B - 2B^{-1}P. \end{aligned}$$

Multiplying both sides of this inequality by nonnegative B^{-1}, we obtain the upper bound

$$A^{-1} \leqslant \mathscr{M}(A)^{-1} - 2B^{-1}PB^{-1}.$$

The lower bound for A^{-1} is derived as follows. We have

$$\begin{aligned} A^{-1}B &= (I - B^{-1}P)^{-1} - 2B^{-1}P\left[I - (B^{-1}P)^2\right]^{-1} \\ &\geqslant (I - B^{-1}P)^{-1} - 2B^{-1}P(I - B^{-1}P)^{-1} \\ &= (I - 2B^{-1}P)(I - B^{-1}P)^{-1} \\ &\overset{(8)}{=} (I - 2B^{-1}P)\mathscr{M}(A)^{-1}B, \end{aligned}$$

implying, since B^{-1} is nonnegative, that

$$A^{-1} \geqslant \mathscr{M}(A)^{-1} - 2B^{-1}P\mathscr{M}(A)^{-1}.$$

Taking into account that by (8)

$$B^{-1}P\mathscr{M}(A)^{-1} = \mathscr{M}(A)^{-1} - B^{-1},$$

we obtain the lower bound

$$A^{-1} \geqslant -\mathscr{M}(A)^{-1} + 2B^{-1},$$

which completes the proof of the theorem. ∎

COROLLARY 1. *If under the hypotheses of Theorem 2 the inverse to the Z-part of A is bounded as*

$$B^{-1} \geqslant C,$$

then

$$-\mathcal{M}(A)^{-1} + 2C \leqslant A^{-1} \leqslant \mathcal{M}(A)^{-1} - 2CPC. \tag{10}$$

In particular, since

$$B^{-1} \geqslant \operatorname{Diag}(B)^{-1} = \operatorname{Diag}(A)^{-1},$$

the above corollary yields

$$\begin{aligned} &-\mathcal{M}(A)^{-1} + 2\operatorname{Diag}(A)^{-1} \\ &\qquad \leqslant A^{-1} \leqslant \mathcal{M}(A)^{-1} - 2\operatorname{Diag}(A)^{-1} P \operatorname{Diag}(A)^{-1}, \end{aligned} \tag{11}$$

providing an especially simple improvement of (2).

By inspection of the proof of Theorem 2 one can easily see that this theorem can be generalized in the following way.

THEOREM 3. *Suppose a matrix $A \in \mathbb{R}^{n\times n}$ can be split into the sum*

$$A = B + P,$$

where B is monotone, $B^{-1}P$ is nonnegative, and $\rho(B^{-1}P) < 1$. Then

$$-(B-P)^{-1} + 2B^{-1} \leqslant A^{-1} \leqslant (B-P)^{-1} - 2B^{-1}PB^{-1}.$$

REMARK 1. Whenever B^{-1} and $B^{-1}P$ are nonnegative, the inequality $\rho(B^{-1}P) < 1$ is equivalent to the monotonicity of $B - P$ [3].

In view of this remark, Theorem 3 implies the following generalization of the Ostrowsky result.

COROLLARY 2. *Under the hypotheses of Theorem 3*

$$|A^{-1}| \leqslant (B-P)^{-1}.$$

We conclude this presentation with a simple example illustrating the bounds (9) and (11) and their relationship with the known bounds (2) and (4). Consider a 3×3 H-matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

with the inverse

$$A^{-1} = \frac{1}{26}\begin{bmatrix} 10 & -4 & -2 \\ 2 & 7 & -3 \\ 4 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 0.384\ldots & -0.153\ldots & -0.0769\ldots \\ 0.0769\ldots & 0.269\ldots & -0.115\ldots \\ 0.153\ldots & 0.0384\ldots & 0.269\ldots \end{bmatrix}.$$

The bounds (9) yield

$$\frac{1}{72}\begin{bmatrix} 0 & -36 & -36 \\ -12 & 3 & -27 \\ -4 & -11 & 3 \end{bmatrix} \leqslant A^{-1} \leqslant \frac{1}{72}\begin{bmatrix} 44 & 4 & 12 \\ 16 & 29 & 3 \\ 20 & 11 & 29 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & -0.5 & -0.5 \\ -0.1(6) & 0.041(6) & -0.375 \\ -0.0(5) & -0.152(7) & 0.041(6) \end{bmatrix}$$

$$\leqslant A^{-1} \leqslant \begin{bmatrix} 0.6(1) & 0.0(5) & 0.1(6) \\ 0.(2) & 0.402(7) & 0.041(6) \\ 0.2(7) & 0.152(7) & 0.402(7) \end{bmatrix},$$

while the simplified bounds (11) provide that

$$\frac{1}{72}\begin{bmatrix} 0 & -36 & -36 \\ -36 & 3 & -27 \\ -36 & -27 & 3 \end{bmatrix} \leqslant A^{-1} \leqslant \frac{1}{72}\begin{bmatrix} 72 & 12 & 12 \\ 36 & 45 & 11 \\ 36 & 27 & 45 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & -0.5 & -0.5 \\ -0.5 & 0.041(6) & -0.375 \\ -0.5 & -0.375 & 0.041(6) \end{bmatrix} \leqslant A^{-1} \leqslant \begin{bmatrix} 1 & 0.1(6) & 0.1(6) \\ 0.5 & 0.625 & 0.152(7) \\ 0.5 & 0.375 & 0.625 \end{bmatrix}.$$

On the other hand, the Ostrowsky bound (2) ensures only that

$$|A^{-1}| \leqslant \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 0.625 & 0.375 \\ 0.5 & 0.375 & 0.625 \end{bmatrix},$$

while its improvement (4) shows that

$$|A^{-1}| \leqslant \frac{1}{182}\begin{bmatrix} 124 & 31 & 35 \\ 38 & 55 & 21 \\ 28 & 7 & 49 \end{bmatrix} = \begin{bmatrix} 0.681\ldots & 0.170\ldots & 0.192\ldots \\ 0.208\ldots & 0.302\ldots & 0.115\ldots \\ 0.153\ldots & 0.0384\ldots & 0.269\ldots \end{bmatrix}.$$

As this example demonstrates, neither (9) nor (4) is sharper than the other. Thus, the new bounds (9) [or (11)] can be combined with the Neumaier bounds (4) to get a sharper result.

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